PICSL : Semi-Lagrangian and Particle Methods for Solving the Vlasov Equation

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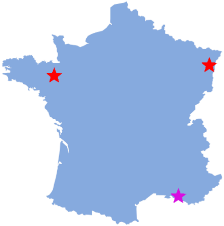


Outline

- What is plasma ?
- How can we model its dynamics ?
- How can we code a simulation in the chosen model ?
- How can we optimize that code ?

Examples of plasma

The fourth state of matter. . . 99% of the universe !

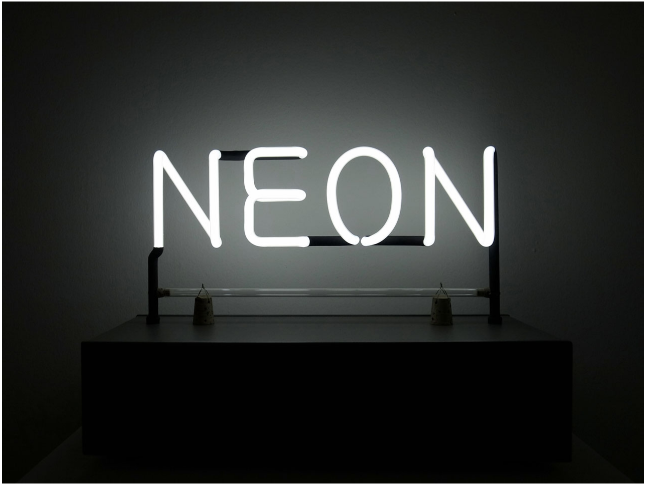
Examples of plasma

The fourth state of matter... 99% of the universe !

lightning

Examples of plasma

The fourth state of matter. . . 99% of the universe !



fluorescent light

Examples of plasma

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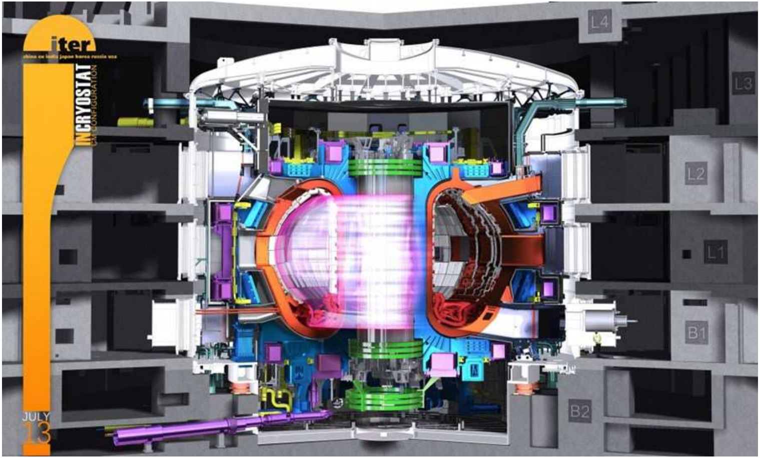

ITER[a] tokamak (controlled thermonuclear fusion)

a. ≪ The way ≫ (in latin) to produce energy

Modelling

$$\begin{cases} \dfrac{\partial f}{\partial t} + \overrightarrow{v} \cdot \dfrac{\partial f}{\partial \overrightarrow{x}} - \dfrac{\mathrm{e}}{m} \overrightarrow{E} \cdot \dfrac{\partial f}{\partial \overrightarrow{v}} = 0 & \text{Vlasov} \\ -\Delta\phi = \dfrac{\rho}{\varepsilon_0} & \text{Poisson} \end{cases}$$

- $f(\overrightarrow{x}, \overrightarrow{v}, t)$: distribution function of the electrons
- $\overrightarrow{E}(\overrightarrow{x}, t) = -\overrightarrow{grad}\,\phi$: the electric field, here self-induced ; ϕ is the associated scalar potential
- ε_0 : vacuum permittivity
- e, m : electron charge and mass
- t : time
- $\overrightarrow{x} \in (\mathbb{R}/(L_x\mathbb{Z})) \times (\mathbb{R}/(L_y\mathbb{Z}))$: particle position (1d, 2d or 3d)
- $\overrightarrow{v} \in \mathbb{R}^2$: particle speed (1d, 2d or 3d)
- $\rho(\overrightarrow{x}, t) = e\left(1 - \displaystyle\int f(\overrightarrow{x}, \overrightarrow{v}, t)d\overrightarrow{v}\right)$: volume charge density

Modelling

Essentially three methods for modelling the particle density inside plasma :

- Semi-Lagrangian methods
- Particle-in-Cell methods
- Eulerian methods

Semi-Lagrangian Methods

- splitting of the Vlasov equation

$$\frac{\partial f}{\partial t} + \overrightarrow{v} \cdot \frac{\partial f}{\partial \overrightarrow{x}} + \frac{q}{m} \overrightarrow{E} \cdot \frac{\partial f}{\partial \overrightarrow{v}} = 0$$

- splitting of the Vlasov equation into two simpler equations :

$$\frac{\partial f}{\partial t} + \overrightarrow{v} \cdot \frac{\partial f}{\partial \overrightarrow{x}} + \frac{q}{m}\overrightarrow{E} \cdot \frac{\partial f}{\partial \overrightarrow{v}} = 0$$

[Cheng and Knorr, 1976]

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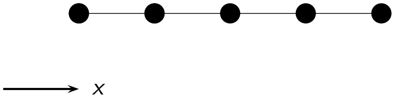
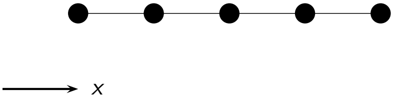


Values after k time steps.



Values after $k+1$ time steps.

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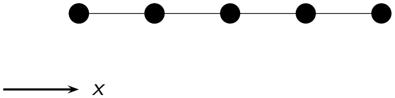

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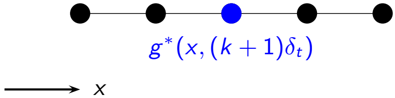

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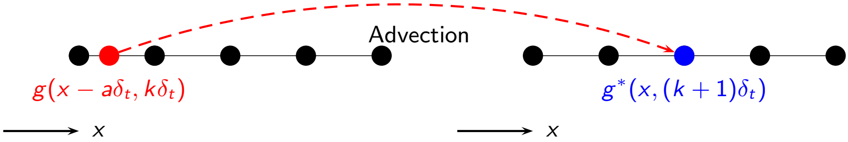

Values after k time steps. Values after $k+1$ time steps.

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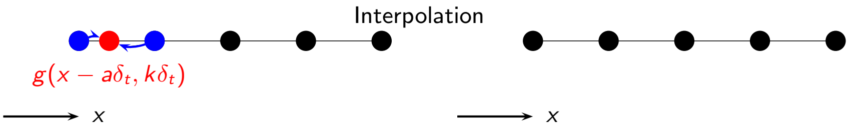

Values after k time steps.

Values after $k+1$ time steps.

- approximation of f via (a lot of) numerical particles
  - one numerical particle represents many real-life particles

Particle-in-Cell Methods

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- particles only interact via the self-induced fields (but don't consider every interaction - it's not N-body model)

- approximation of f via (a lot of) numerical particles
  - one numerical particle represents many real-life particles
- particles only interact via the self-induced fields (but don't consider every interaction - it's not N-body model)
- $f(\overrightarrow{x}, \overrightarrow{v}, t) = \sum_{k=1}^{N} weight_k \delta(\overrightarrow{x} - \overrightarrow{x_k})\delta(\overrightarrow{v} - \overrightarrow{v_k})$
  - δ is the distribution of Dirac :
  - $\int_{\mathbb{R}} \delta(x)dx = 1$
  - $\delta(0) = +\infty$
  - $\delta(x) = 0$ when $x \neq 0$

Comparison PIC / SL Methods

- SL
  - \+ only stores f via a grid on positions and speeds : faster on 1D (2D grid) and 2D (4D grid)
  - \- slower on 3D (6D grid is too much)
- PIC
  - \+ only stores a grid for the fields on positions : faster on 3D
  - \- also stores an array of particles : slower on 1D and 2D
  - \- requires a lot of particles : stochastic convergence in $\frac{1}{\sqrt{N}}$

Dispersion relation

Principle : solve exactly the linearized equation.

If $f^0 \equiv f^0(v)$ is an equilibrium solution and
$f(t=0,x,v) = f^0(v) + A\widehat{f}(0,v)e^{ik.x}$ where $A \ll 1$ then

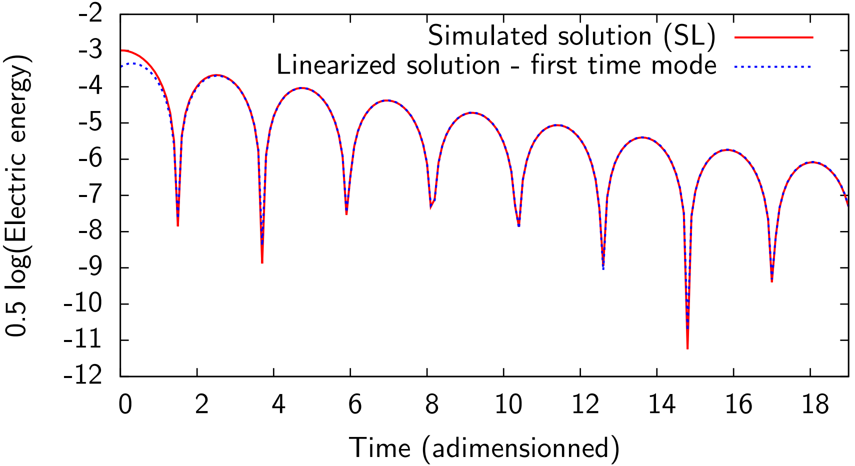
$$E(t,x) = Ae^{ik.x} \sum_{\omega \in D^{-1}(\{0\})} Res(\omega)e^{-i\omega t}\frac{k}{|k|} + O(A^2),$$

with D an analytic function depending only on f^0 and k and Res an analytic function depending on f^0, k and $\widehat{f}(0,.)$.

- $\forall \omega \in D^{-1}(\{0\}), \mathcal{I}\mathrm{m}(\omega) \leq 0 \Rightarrow$ stable. e.g. $f^0 = \dfrac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}}$.
- $\exists \omega \in D^{-1}(\{0\}), \mathcal{I}\mathrm{m}(\omega) > 0 \Rightarrow$ unstable. e.g. $f^0 = \dfrac{v^2 e^{-\frac{v^2}{2}}}{\sqrt{2\pi}}$.

Test case : Landau

$$f(0,x,v) = \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}}(1 + A\cos(kx)) \Rightarrow \forall\omega \in D^{-1}(\{0\}), \mathcal{I}\mathrm{m}(\omega) \leq 0$$



Test case : $2d \times 2d$

Problem : linearly, $2d$ solution is a superposition of $1d$ solutions.
Solution : find the term in A^2 in the expansion of f.

Principle : if $f = f^0(v) + Af^1(t,x,v) + A^2 f^2(t,x,v)$ and $E = AE^1(t,x) + A^2E^2(t,x)$ then

$$\left\{ \begin{array}{l} \partial_t f^2 + v.\nabla_x f^2 - E^2.\nabla_v f^0 - E^1.\nabla_v f^1 = 0, \\ -\Delta_x \Phi^2 = -\int_{\mathbb{R}^2} f^2 \, \mathrm{dv}, \\ E^2 = -\nabla_x \Phi^2. \end{array} \right.$$

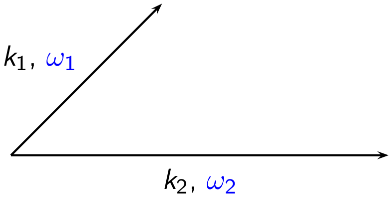
It is the linearized equation but with a source term $E^1.\nabla_v f^1$ that is given by the linear analysis. The solution is given by the Duhamel's formula.

Consequence :

- one can deduce the dominant time mode of f^2.

Test case : $2d \times 2d$

$$f(t=0,x,v) = f^0(v) + A\alpha(v)e^{ik_1.x} + A\beta(v)e^{ik_2.x}$$



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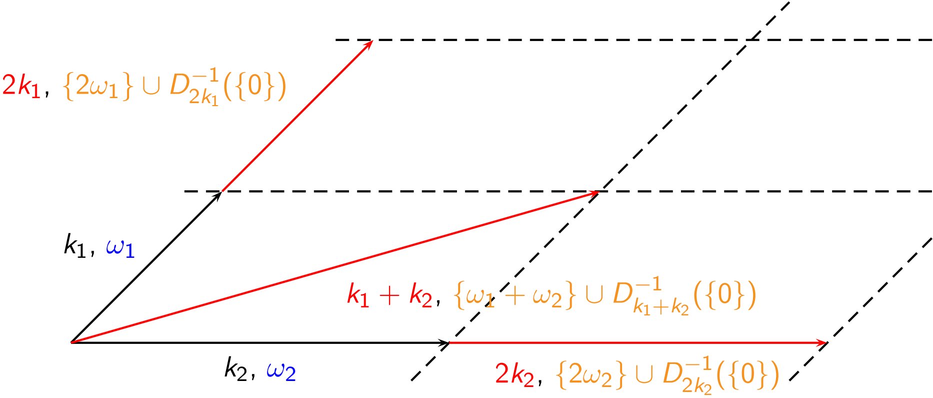

Test case : $2d \times 2d$

If $f^0(v) = \frac{v_x^2 e^{-\frac{|v|^2}{2}}}{2\pi}$ and $L = 4\pi$ then for the spatial modes :
mode k is unstable $\Leftrightarrow D_k^{-1}(\{0\}) \cap (\mathbb{R} + i\mathbb{R}_+^*) \neq \emptyset \Leftrightarrow k = \pm\frac{1}{2}(1,0)$
We take $f(0,x,v) = (1 + Acos(\frac{y}{2}) + Acos(\frac{x+y}{2}))f^0(v)$, $A = 0.001$.
The theory makes us expect :

- a Landau damping at the order 1 in A
- an explosion of the space mode $\frac{1}{2}(1,0)$ at the order 2

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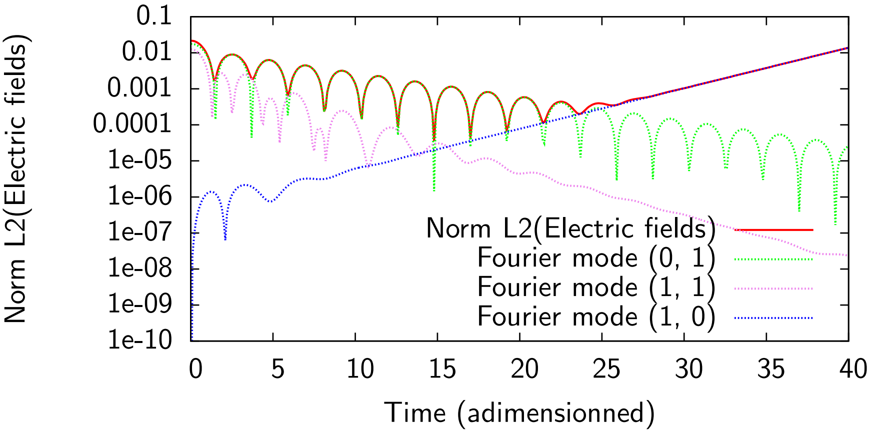


The movie

http://www.barsamian.am/Slides/2dx2d_rho.mpg

Test case : Badsi Herda

Two species, $\epsilon = \sqrt{\frac{m_i}{m_e}}$.

Scaling

$$\begin{cases} \partial_t f_i + v.\partial_x f_i + E.\partial_v f_i = 0, \\ \partial_t f_e + \frac{1}{\epsilon} v.\partial_x f_e - \frac{1}{\epsilon} E.\partial_v f_e = 0, \\ \partial_x E = \int_{\mathbb{R}^2} f_i - f_e \,\mathrm{dv}\,. \end{cases}$$

Initial conditions and initialisation :

$$\begin{cases} f_e(t=0,x,v) = \dfrac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}}, \\ f_i(t=0,x,v) = 8\dfrac{e^{-2v^2}}{\sqrt{2\pi}}(1 + A\cos(kx)). \end{cases}$$

Time modes :

$$D \equiv D_1(\xi) + D_2(\epsilon\xi)$$

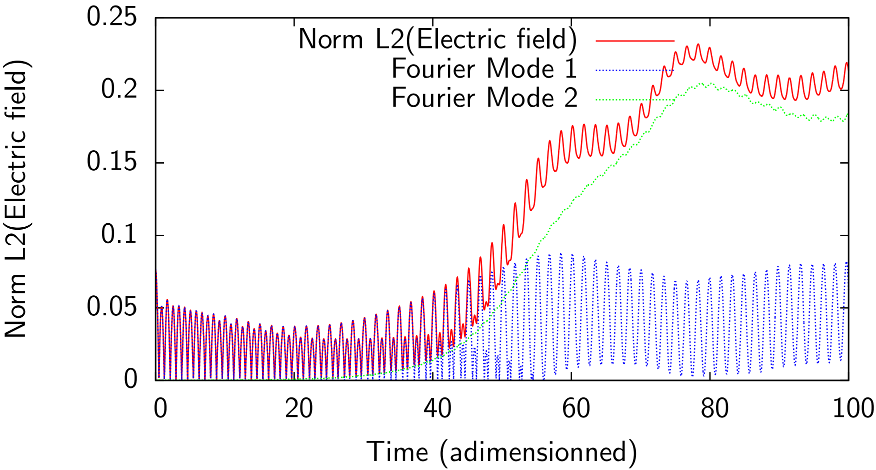
When $\epsilon \to 0$, one has

$$D^{-1}(\{0\}) \equiv [D_1 + D_2(0)]^{-1}(\{0\}) \sqcup \frac{1}{\epsilon} D_2^{-1}(\{0\}) + O(\epsilon).$$

Test case : Badsi Herda

With $\omega_1 \in i\mathbb{R}_+^*$ and $\mathcal{I}m(\omega_2) < 0$: $E \equiv Ae^{ikx}(\alpha e^{-i\omega_1 t} + \beta e^{-i\frac{\omega_2}{\epsilon}t})$

Numerically, with $\epsilon = \sqrt{0.1}$ and $A = 0.01$, we have :



Problem : only the mode $2k$ explodes.

Test case : Badsi Herda

With bilinear analysis, one expects

$$E \equiv Ae^{ikx}(\alpha e^{-i\omega_1 t} + \beta e^{-i\frac{\omega_2}{\epsilon}t}) + A^2\gamma e^{2ikx}e^{-i2\omega_1 t}.$$

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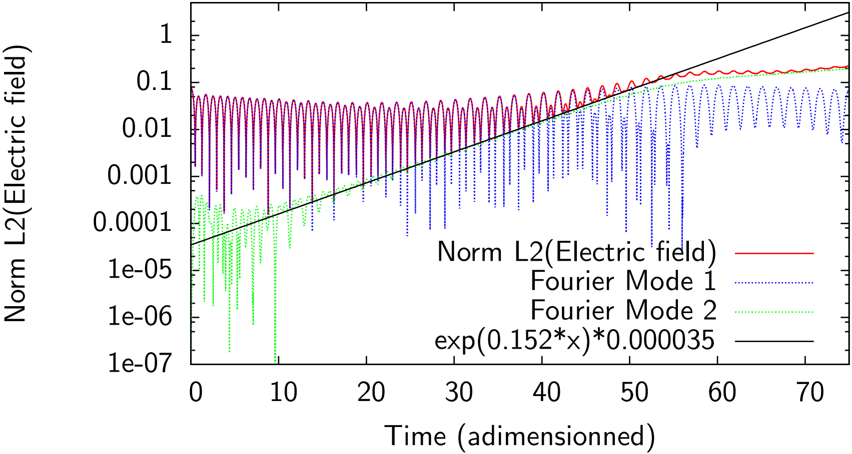
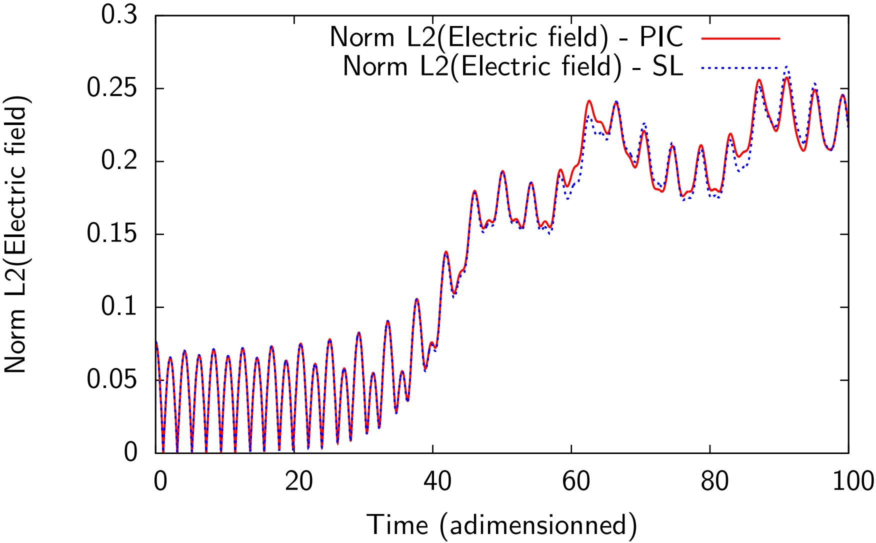


Numerical comparison PIC / SL (1D)



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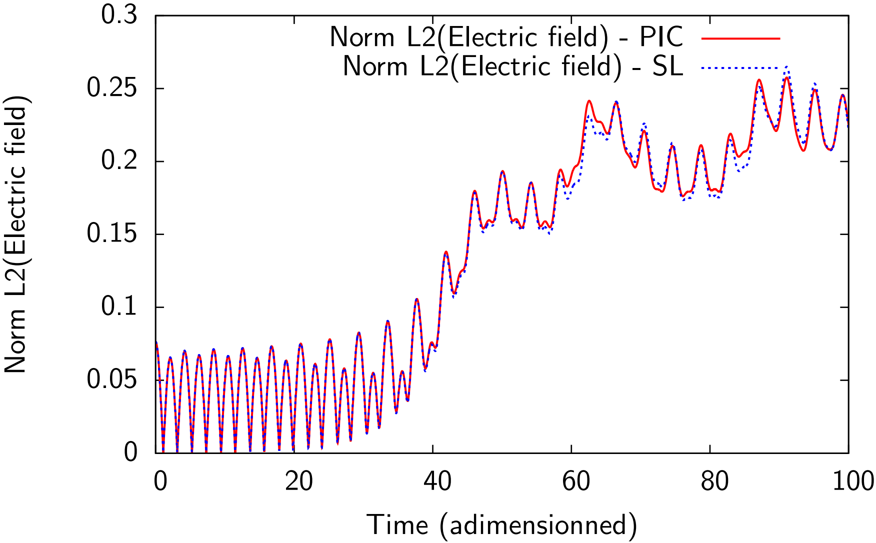

SL : grid 128×256, 2.5h ; PIC : 8.192×10^9 particles, 1350h

Future Work

- extension of the Vlasov-Poisson model to the Vlasov-Maxwell model
- adding of an external magnetic field
- more precise modelling (drift-kinetic)


That's all Folks!